The Ising model

There is no exact solution to the dipole model. This is why, in 1920, Lenz proposed to symplify the problem in the form

$$H = -J \sum_{\langle i,j \rangle} s_i s_j - h \sum_i s_i$$

where $s_i = \pm 1 \quad \forall i$

He asked Ernst Ising, his PhD student, to try and find solutions to it. We are going to look at them and see what we learn.

Mean-field solution

Let's take the product $s_i s_j$, which is one of the causes of the non-solvability of the Ising model (at least, not easy solvability). We can rewrite it as:

$$s_i s_j = (s_i - \langle s_i \rangle)(s_j - \langle s_j \rangle) + \langle s_i \rangle s_j + \langle s_j \rangle s_i - \langle s_i \rangle \langle s_j \rangle$$

and then claim that the fluctuations around the averages are negligible. Thus

$$s_i s_j \simeq \langle s_i \rangle s_j + \langle s_j \rangle s_i - \langle s_i \rangle \langle s_j \rangle$$

We insert this in the calculation of the partition function:

$$Z_N = \sum_{\{s_i\}} e^{\beta J \sum_{\langle i,j\rangle} s_i s_j + \beta h \sum_i s_i} \simeq$$

Lattice of N spins

$\hookrightarrow s_1 = \pm 1, \; s_2 = \pm 1, \; \dots \; s_N = \pm 1$

$m = \langle s_i \rangle \;\; \forall i$

$$\simeq \sum_{\{s_i\}} e^{-\beta J \sum_{\langle i,j\rangle} m^2 + \beta J \sum_{\langle i,j\rangle} m(s_i + s_j) + \beta h \sum_i s_i} =$$

NOTE: $\sum_{\langle i,j\rangle} m^2 = \frac{z}{2} N m^2$ — coordination number

$$= e^{-\beta J \frac{z}{2} N m^2} \sum_{\{s_i\}} e^{\beta J z m \sum_i s_i + \beta h \sum_i s_i} =$$

$$= e^{-\beta J \frac{z}{2} N m^2} \sum_{\{s_i\}} \prod_i e^{\beta (J z m + h) s_i} =$$

$$= \boxed{2^N e^{-\beta J \frac{z}{2} N m^2} \cosh\left[\beta (J z m + h)\right]^N = Z_N}$$

Then the free energy is

$$F_N = -k_B T \ln Z_N = -k_B T N \left\{ \left[-\beta J \frac{z}{2} m^2 + \ln 2 + \ln \cosh\left[\beta (J z m + h)\right] \right] \right\}$$

extensive

Let's find the magnetization

$$m = -\frac{1}{N}\frac{\partial F_N}{\partial h} = -\frac{1}{N}\frac{\partial}{\partial h}\left[-k_B T N \ln \cosh[\beta(Jzm+h)]\right] =$$

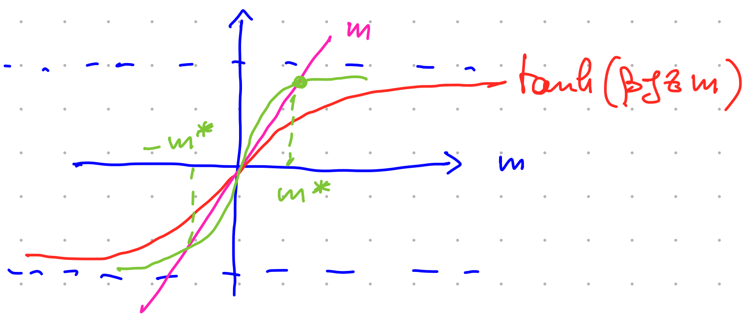
$$= k_B T \, \frac{\sinh[\beta(Jzm+h)]}{\cosh[\beta(Jzm+h)]}\,\beta = \boxed{\tanh[\beta(Jzm+h)] = m}$$

case $h=0$ (spontaneous magnetization; case $h \neq 0$ similar to Curie-Weiss)

The slope at the origin is

$$\tanh(x) \simeq x - \frac{1}{3}x^3$$

$$\Rightarrow \quad \tanh(\beta J z m) \simeq \beta J z m$$



- If $\beta J z < 1$ only intersection is in $0 \Rightarrow$ no magnetization if $T > T_c = \frac{Jz}{k_B}$

- If $\beta J z > 1$ there is a second intersection m^* $(T < T_c)$

m^* (and $-m^*$ by symmetry) is stable under iterative solution. We are soon going to see that it is thermodynamically stable.

Sanity check: $\quad m \simeq 0$, $T \simeq T_c$

$$m = \tanh(\beta J z m) \simeq \beta J z m - \frac{1}{3}(\beta J z m)^3$$

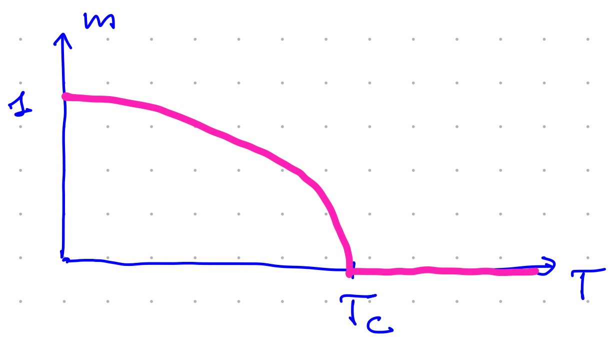
$$\Rightarrow 3(1-\beta J z) = -\beta J z (\beta J z m)^2$$

Then

$$\frac{3(\beta J z - 1)}{\beta J z} = (\beta J z\, m)^2$$

$$\Rightarrow \quad m = \sqrt{\frac{3}{(\beta J z)^3}} \left(\frac{T_c - T}{T}\right)^{1/2} \underset{\beta=\beta_c}{\simeq} \sqrt{3}\, t^{1/2}$$

as for Curie-Weiss!



Now we move back to the free-energy:

$$F_N = -k_B T N \left\{ -\beta \frac{Jz}{2} m^2 + \ln 2 + \ln \cosh\left[\beta(Jzm + h)\right]\right.$$

Let's Taylor expand it for m small (close to T_c, $h \sim 0$)

$$F_N \sim -k_B T N \left\{ -\beta \frac{Jz}{2} m^2 + \ln 2 + \ln\left[1 + \frac{1}{2}\beta^2 (Jzm + h)^2 + \right.\right.$$

$$\left.\left. + \frac{1}{4!}\beta^4 (Jzm + h)^4\right]\right\} \simeq$$

$$\simeq -k_B T N \left\{ -\beta \frac{Jz}{2} m^2 + \ln 2 + \frac{1}{2}\beta^2 (Jzm + h)^2 + \right.$$

$$\left. + \frac{1}{4!}\beta^4 (Jzm + h)^4 - \frac{1}{2}\frac{1}{2^2}\beta^4 (Jzm + h)^4 \right\} =$$

$$= -k_B T N \left\{ -\beta \frac{Jz}{2} m^2 + \ln 2 + \frac{1}{2}\beta^2 (Jzm + h)^2 + \right.$$

$$\left. + \frac{1}{4!}\beta^4 (Jzm + h)^4 - \frac{1}{2}\frac{1}{2^2}\beta^4 (Jzm + h)^4 \right\} =$$

does not depend on m: irrelevant

$$= N\left\{ \frac{Jz}{2} m^2 - k_B T \ln 2 - \frac{Jz}{2}\beta Jz m^2 - \beta Jz h m - \frac{1}{2}\beta h^2 + \right.$$

$$\left. - \frac{1}{4!}\beta^3 (Jzm)^4 + \frac{1}{8}\beta^3 (Jzm)^4 \right\} =$$

$$= N\left\{ \frac{Jz}{2}(1 - \beta Jz) m^2 - \beta Jz h\, m + \frac{1}{12} Jz (\beta Jz)^3 m^4 \right\}$$

At last:

$$F_N = N Jz \left\{ \frac{1}{2}\left(\frac{T - T_c}{T_c}\right) m^2 + \frac{1}{12}\underbrace{\left(\frac{T_c}{T_c}\right)^3}_{\simeq 1} m^4 - \left(\frac{T_c}{T_c}\right) m h - \frac{T}{T_c}\ln 2 \right\}$$

because $T \simeq T_c$

At very last:

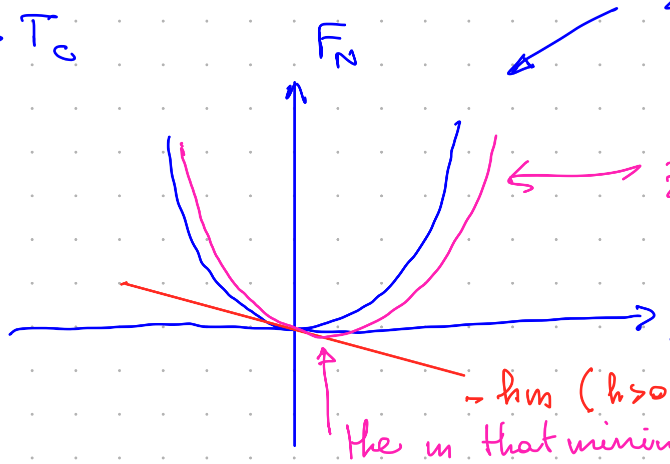
overall irrelevant but a good reminder that at $T \to \infty$ only entropy matters

$$F_N = N \underbrace{k_B T_c}_{\text{natural energy scale for the system close to } T_c} \left\{ \frac{1}{2}\left(\frac{T - T_c}{T_c}\right) m^2 + \frac{1}{12} m^4 - m h - \frac{T}{T_c}\ln 2 \right\}$$

Let's study it graphically:

- $T > T_c$

$\frac{1}{2}\frac{\Delta T}{T_c} m^2 + \frac{1}{12} m^4$



$\frac{1}{2}\frac{\Delta T}{T_c} m^2 + \frac{1}{12} m^4 - hm$

$-hm$ ($h>0$, otherwise in the other direction)

the m that minimizes the free energy is shifted to the right

$$\frac{\partial F_N}{\partial m} = 0$$

to search for the equilibrium m

$T > T_c \Rightarrow \Delta T > 0$

$$\frac{\partial F_N}{\partial m} = k_B T_c N \left\{ \frac{\Delta T}{T_c} m - h + \frac{1}{3} m^3 \right\} = 0$$

m small (close to T_c) so m^3 is negligible.

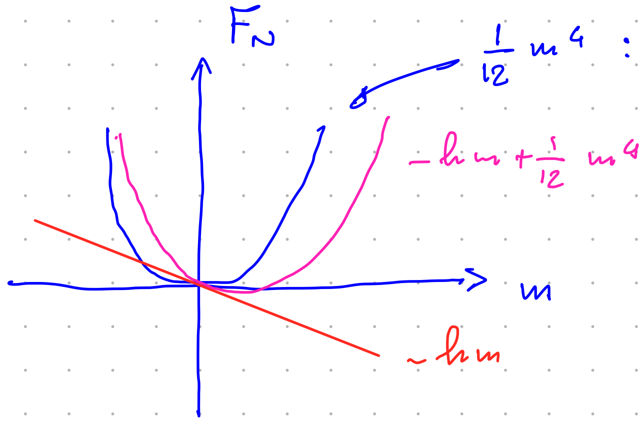
$$\Rightarrow \quad m \simeq \frac{T_c}{\Delta T} h$$

[no worries about $\Delta T = 0$: $T = T_c$ has a special treatment]

and $m = 0$ if $h = 0$

• $T = T_c$

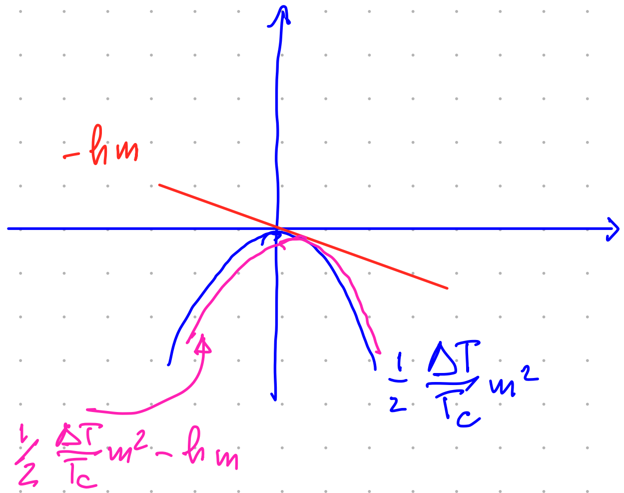
$$F_N = k_B T_c N \left\{ -hm + \frac{1}{12} m^4 \right\}$$



just like in Curie-Weiss!

$$\frac{\partial F_N}{\partial m} = -h + \frac{1}{3} m^3 \quad \Longrightarrow \quad m \sim 3^{1/3} h^{1/3}$$

• $T < T_c$



$(T < T_c \ : \ \Delta T < 0)$

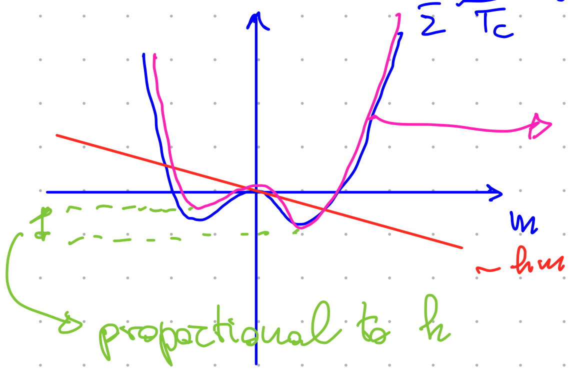
If we neglect higher order terms (m^4) then

$$\frac{\partial F_N}{\partial m} = k_B T_c N \left\{ -h + \frac{\Delta T}{T_c} m \right\} = 0 \qquad m = \frac{T_c}{\Delta T} h$$

but it is a maximum of the free energy!

The minima of the free energy are for $m = \pm \infty$: non-physical

We must thus include m^4



$$\frac{\partial F_N}{\partial m} = k_B T_c N \left\{ \frac{\Delta T}{T_c} m + \frac{1}{3} m^3 - h \right\} = 0$$

Let's start from $h=0$

$$\frac{\Delta T}{T_c} m + \frac{1}{3} m^3 = 0 \implies \begin{cases} m = 0 \\ m^* = \pm\sqrt{3}\left(-\frac{\Delta T}{T_c}\right)^{1/2} \end{cases}$$

$\Delta T < 0$

non-analytic

What about the dependence on h?

Let's work perturbatively (since h is small):

$m = m^* + \alpha h$ where α is a coefficient.

Then

$$\frac{\partial F_N}{\partial m} = N k_B T_c \left\{ \frac{\Delta T}{T_c}(m^* + \alpha h) + \frac{1}{3}\left(m^{*3} + 3 m^{*2}\alpha h \right) - h \right\} = 0$$

neglect higher order terms

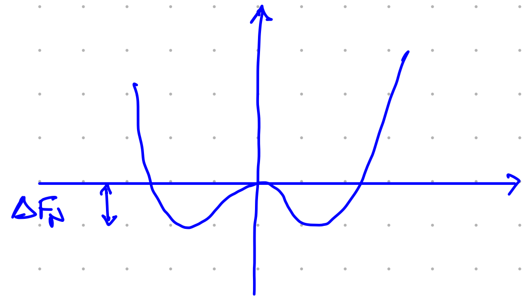
$$\Rightarrow \underbrace{\frac{\Delta T}{T_c} m^* + \frac{1}{3} m^{*3}}_{=0 \text{ because } m^* \text{ satisfies this equation}} + \alpha h \left(\frac{\Delta T}{T_c} + 3\frac{|\Delta T|}{T_c} \right) - h = 0$$

$$\Rightarrow \alpha = \frac{1}{\frac{\Delta T}{T_c} + 3\frac{|\Delta T|}{T_c}} > 0$$

Thus m^* is changed proportionally to h

Careful: this holds if $m^* \neq 0$ otherwise the expansion is the one that directly gives $h^{1/3}$

Now let's go back to the free energy:



$\boxed{\Delta F_N \propto N}$, which implies that it is finite

Is any of the two states stable? Infinitely so?

Intuitively, the dynamical switch between the two takes place over a timescale

$$\tau = \tau_0\, e^{\Delta F_N / k_B T}$$

proportional to N

Any finite system will switch if we wait long enough!!!

Only an infinite system is going to reside infinitely long in one state.

We are talking here of:

1) spontaneous symmetry breaking:
there is no reason for the system to choose one state over the other: infinitesimal fluctuations, small h

2) thermodynamic limit ($N \to \infty$): symmetry breaking is stabilized (also: ergodicity breaking: not all states are visited anymore)